

# Orthomodular Lattices and Quantales<sup>1</sup>

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Let  $L$  be a complete orthomodular lattice. There is a one to one correspondence between complete Boolean subalgebras of  $L$  contained in the center of  $L$  and endomorphisms  $j$  of  $L$  satisfying the Borceux–van den Bossche conditions.

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## 1. INTRODUCTION

In Román and Zuazua (2005), we study the notion of an idempotent right-sided quantale versus the concept of orthomodular lattice. We proved that a complete orthomodular lattice  $L$  has a natural structure of an idempotent right-sided quantale if we take the central cover of an arbitrary element  $a$  (denoted by  $e(a)$ ) of  $L$ ; this construction induces an endomorphism of  $L$  satisfying certain conditions, see section 1 for more details. Therefore, there is a natural question after this claim: which endomorphisms of  $L$  can produce an idempotent, right-sided quantale?

Recall that when the endomorphism  $j : L \rightarrow L$  satisfies the Borceux–van den Bossche’s (denoted by B.-V.B.) conditions, then  $j$  induces an idempotent, right-sided quantale in  $L$ ; in fact, the structure is quite simple: if  $a, b$  are arbitrary elements of  $Q$  then  $a \& b = a \wedge j(b)$ . For details see Borceux and van den Bossche (1986) or Román and Zuazua (2005). The endomorphism  $j$  is a closure operator satisfying three conditions which are related with the concepts of nucleus and quantic nucleus considered in intuitionistic logic and quantum logic. In Román and Rumbos (1991), we gave a characterization of nuclei in orthomodular lattices and quantic lattices which in particular produces a characterization of quantic nuclei in orthomodular lattices. The difference here is the binary operation  $\&^F$  considered in Román and Rumbos (1991). This operation was first considered by

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Finch (1970), where he suggested that  $\&^F$  has very similar properties with the connectives  $(\wedge, \rightarrow)$ , considered in classical logic and intuitionistic logic. In fact, this is true but the main difference is the lack of the associativity property of  $\&^F$ . Nevertheless,  $-\&^F b : L \rightarrow L$  has a right adjoint which is the Sasaki hook, for an arbitrary element  $b$  of  $L$ . Unfortunately, the operation  $b\&^F - : L \rightarrow L$  does not have a nice property such as  $\&^F$  is associative,  $b\&^F -$  has a right adjoint or  $b\&^F -$  preserves order. In fact, if  $L$  satisfies one of these conditions then  $L$  will be a Boolean algebra and conversely, if  $L$  is a Boolean algebra then  $a\&^F b$  is just  $a \wedge b$ . For details see Román and Rumbos (1991).

After all these comments, I would like to say some words about the result I will prove. Last year, Prof. R. Greechie suggested the following idea: perhaps the work of M. F. Janowitz can help to characterize all the endomorphisms of a complete orthomodular lattice satisfying the B.-V.B. conditions; I must say he was quite right, the paper of M. F. Janowitz, under the title “Residuated Closure Operators” gave me some ideas that are crucial in the proof of the main result of this article (see Janowitz, 1967).

Finally, the reader must note that by “the logic of quantum mechanics” we mean the lattice theoretic “quantum logics” of Birkhoff and von Neumann (1936), hence we do not consider the quantales introduced by Yetter (1990), under the name of “Girard Quantales” where he considers a different logic for quantum mechanics; roughly speaking the logic considered by David Yetter is a logic involving an associative (in general noncommutative) operation “and then.” Yet, Girard quantales have a close relation with linear logic, a logic introduced by J. Y. Girard, the reader can consult (Finch, 1970) for details and comments about this logic. Clearly, a natural question is if linear logic has some relation with the quantum logic introduced by Birkhoff and von Neumann, we will look about this problem in a future work.

The article is organized as follows. In the first section we introduce the concepts we need for our purposes. In the second section we show the main results of this article. We prove that if we have an arbitrary  $j : L \rightarrow L$  satisfying the Borceux–van den Bossche conditions then  $j(L)$  is a Boolean subalgebra of  $L$  contained in the center of  $L$  and conversely every Boolean subalgebra  $M$  of  $L$  contained in the center of  $L$  induces an endomorphism  $k$  satisfying the B.-V.B. conditions. Hence there is a one to one correspondence between endomorphisms  $j$  satisfying the B.-V.B. conditions and Boolean subalgebras of  $L$  contained in the center of  $L$ .

## 2. SECTION I

*Definition 1.* A quantale  $Q$  is a lattice having arbitrary joins  $\vee$  together with an associative product  $\&$  such that

1.  $a \& (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \& b_i)$ ;
2.  $(\bigvee_{i \in I} a_i) \& b = \bigvee_{i \in I} (a_i \& b)$

for all  $a, b, a_i, b_i \in Q$ .

Moreover, we will say that the quantale  $Q$  is an idempotent and right-sided if it satisfies the following two conditions

1.  $a \& 1 = a$ ;
2.  $a \& a = a$  for all  $a \in Q$ .

*Remark 1.* In Borceux and Van Den Bossche (1986) F.Borceux and G. van den Bossche proved that given any complete lattice  $(Q, \leq)$  there is a one-to-one correspondence between binary operations  $\& : Q \times Q \rightarrow Q$  making  $Q$  an idempotent, right-sided quantale and closure operations  $j : Q \rightarrow Q$  satisfying the following axioms (these are the B.-V.B. conditions):

1.  $a \leq j(a)$ ;
2.  $j(a \wedge j(b)) = j(a) \wedge j(b)$ ;
3.  $a \wedge j(\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge j(b_i))$ ;
4.  $(\bigvee_{i \in I} a_i) \wedge j(b) = \bigvee_{i \in I} (a_i \wedge j(b))$ .

We just mention a simple consequence of this result. If we have an endomorphism  $j$  of a complete lattice  $Q$  satisfying the B.-V.B. conditions and we take the fixed points of  $j$  (denoted by  $Q_j$ ) then  $Q_j$  is not only a quantale is a locale as the reader can check easily. Hence, in logical terms any idempotent and right-sided quantale  $Q$  has a locale as a sublattice; i.e.,  $Q$  has a model of intuitionistic logic.

In the rest of this article we shall work only with an idempotent right-sided quantale. There are many examples of these quantales. For instance, any Locale  $H$  is an idempotent, right-sided quantale in a trivial way; the binary operation  $\&$  is just  $\wedge$ . The closed ideals of a  $C^*$ -algebra is also an example, here the binary operation  $\&$  is just the closure of the product of two ideals. For the next example we need some definitions. We begin with the following

*Definition 2.* A complete lattice  $L = (L, \vee, \wedge, \perp)$  is a complete orthomodular lattice if there exists a unary operation  $\perp : L \rightarrow L$  satisfying the conditions:

1.  $a^{\perp\perp} = a$ .
2.  $(\bigvee_{i \in I} a_i)^{\perp} = \bigwedge_{i \in I} a_i^{\perp}$ , for all  $a, a_i \in L$  and any set  $I$
3.  $a \vee a^{\perp} = 1$ .
4.  $a \wedge a^{\perp} = 0$ .

Moreover,  $L$  satisfies the following weak modularity property:

Given any  $a, b \in L$  with  $a \leq b$  then  $b = a \vee (a^{\perp} \wedge b)$  (equivalently  $a = (a \vee b^{\perp}) \wedge b$ ).

As we said in the introduction, if we have a complete orthomodular lattice  $L$ , the way of inducing an idempotent, right-sided quantale structure in  $L$  is by taking the *central cover of an element*. We shall introduce more concepts.

First of all, given two arbitrary elements  $a, b$  of an arbitrary complete orthomodular lattice  $L$ ,  $a \&^F b = (a \vee b^\perp) \wedge b$  and the Sasaky hook is given by the following rule:  $a \rightarrow b = (a \wedge b) \vee a^\perp$ . Since we are interested in the center of a complete orthomodular lattice, we consider first the notion of compatibility:

*Definition 3.* Let  $L$  be a complete orthomodular lattice. We say  $a, b \in L$  are compatible elements (denoted by  $bCa$ ) if and only if  $b \&^F a = a \wedge b$ .

Notice that whenever  $a, b$  are compatible elements it is easy to see that  $a \&^F b = a \wedge b$  also holds.

The simplest example of a pair of elements  $a, b$  which are compatible is whenever one of these elements belongs to the center  $Z(L)$  of the complete orthomodular lattice  $L$ . The definition of the center is as follows:

*Definition 4.* Let  $L$  be an arbitrary complete orthomodular lattice. The center of  $L$ , denoted by  $Z_F(L)$  is the set

$$Z_F(L) = \{a \in L \mid a \&^F b = b \&^F a = a \wedge b \text{ for all } b \in L\}$$

Notice that  $Z_F(L)$  is a Boolean subalgebra of  $L$ . In particular,  $0, 1$  belong always to the center of  $L$ . Hence  $Z_F(L)$  is nonempty. We define now the central cover of an arbitrary element of  $L$ .

*Definition 5.* Let  $L$  be a complete orthomodular lattice. If  $a$  is an arbitrary element of  $L$ , the central cover of  $a$  is given by

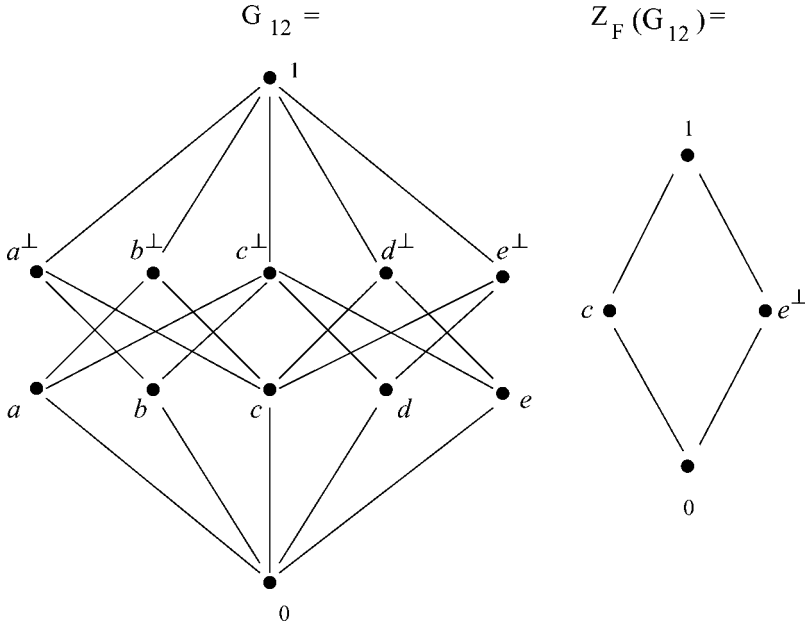
$$e(a) = \wedge \{z \in Z_F(L) \mid a \leq z\}.$$

The central cover of an element always exists since  $1$  is an element of this set. The reader can see the comments in Beltrametti and Cassinelli (1981, p. 129). The next proposition give us the example we are interested in.

**Proposition 6.** (Román and Zuazua, 2005) *The map  $e : L \rightarrow L$  satisfies the Borceux–van Den Bossche conditions.  $L$  has a binary operation  $\&$ , defined by  $a \& b = a \wedge e(b)$ , making it an idempotent, right-sided quantale.*

Therefore, this is the first example of an endomorphism  $j$  of  $L$  making it an idempotent, right-sided quantale. Clearly, the orthomodular lattice must be complete if one wants to preserve the definition of a quantale. However, almost all the concepts described above, can be defined in an arbitrary orthomodular lattice. If we take the classical example of the closed subspaces of a Hilbert space  $H$ , the center is trivial; the only elements of the center are  $0, 1$ . In the literature an

orthomodular lattice is called irreducible whenever it has trivial center. Clearly, if we have an irreducible orthomodular lattice the quantale structure that we get is not really interesting. However, this phenomenon does not occur always. We just mention one example of a finite orthomodular lattice with nontrivial center. Namely,  $G_{12}$ .



We close this section with another comment. The binary operation  $\&^F$  is really important for the construction of the second binary operation  $\&$ ; in fact, all the concepts we had were defined in terms of  $\&^F$ , despite the lack of associativity or equivalently that the endomorphism  $a\&^F -$  does not necessarily preserves order.

## 2. SECTION II

We shall prove now the main result of this article. We will assume  $L$  is an arbitrary complete orthomodular lattice. If we take an arbitrary endomorphism  $j$  of  $L$  satisfying the B.-B.V. conditions then clearly  $L$  is an idempotent, right-sided quantale. We just define  $a\&b = a \wedge j(b)$ , for elements  $a, b$  in  $L$ . Now, the question is “which endomorphisms  $j$  of  $L$  satisfy the B.-V.B. conditions?” Actually, can we give a characterization of such endomorphisms in terms of another concept? We shall see that this is the case. We would like to recall that some of the results are inspired by the work of Janowitz (1967). First of all, we begin with the

following:

**Lemma 7.** *If  $j : L \rightarrow L$  is an arbitrary endomorphism of a complete orthomodular lattice, then given any element  $a$  of  $L$ ,  $j(a)$  satisfies the following identity.*

$$j(a) = \wedge \{x \in L \mid j(x) = x, a \leq x\}.$$

**Proof:** Indeed, let us call  $z$  the RHS of the last equality. Since,  $a \leq j(a)$  and  $j$  is idempotent, we have  $z \leq j(a)$ . Now,  $a \leq z$  and since  $j$  preserves order we get  $j(a) \leq j(z) = z$ . Hence,  $z = j(a)$ . As we claimed.  $\square$

We must notice that  $j(0)$  is equal to 0. The reason is quite simple, just take the empty set for  $I$  in the third property of B.-V.B. conditions. We now prove the next

**Proposition 8.** *Suppose  $L$  is an arbitrary complete orthomodular lattice and  $j$  is an endomorphism satisfying the B.-V.B. conditions then the subset  $L_j$  of  $L$  defined by*

$$L_j = \{x \in L \mid j(x) = x\}.$$

*is a complete Boolean sublattice of  $L$  contained in the center of  $L$ .*

**Proof:** We check first,  $L_j$  is a complete lattice. If  $\{b_i\}_{i \in I}$  is an arbitrary family of elements of  $L_j$  and taking  $a = 1$ , by the third property of the B.-V.B. conditions we have

$$j(\bigvee_{i \in I} b_i) = 1 \wedge j(\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (1 \wedge j(b_i)) = \bigvee_{i \in I} j(b_i) = \bigvee_{i \in I} b_i.$$

Hence,  $L_j$  is a complete lattice. We shall see now  $L_j$  is closed under complements; i.e., if  $a \in L_j$  then  $a^\perp$  also is an element of  $L_j$ . We already knew  $a^\perp \leq j(a^\perp)$ . We only need to check:  $j(a^\perp) \leq a^\perp$ . This is equivalent to:  $j(a^\perp) \&^F a \leq 0$ . Since  $-\&^F a$  has a right adjoint, namely the Sasaki hook, it is enough to check this. We calculate  $j(a^\perp) \&^F a$ .

$$j(a^\perp) \&^F a = (j(a^\perp) \vee a^\perp) \wedge a = a \wedge j(a^\perp).$$

Since  $a^\perp \leq j(a^\perp)$ . Now, since  $a \in L_j$ , we shall see,  $a \wedge j(a^\perp) = 0$ .

$$\begin{aligned} a \wedge j(a^\perp) &= j(a) \wedge j(a^\perp) = j(a^\perp \wedge j(a)) \leq j(a^\perp \&^F j(a)) = j(a^\perp \& a) \\ &= j(0) = 0 \end{aligned}$$

Hence,  $a \wedge j(a^\perp) = 0$  and  $j(a^\perp) \leq a^\perp$ . Therefore  $L_j$  is closed under complements.

We check now  $L_j$  is a Boolean sublattice of  $L$ . Indeed, if  $a, \{b_i\}_{i \in I}$  belong to  $L_j$ , from the third property of the B-V.B. conditions we have

$$a \wedge (\bigvee_{i \in I} b_i) = a \wedge j(\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge j(b_i)) = \bigvee_{i \in I} (a \wedge b_i).$$

Therefore  $L_j$  is a distributive lattice closed under complements; i.e.,  $L_j$  is a Boolean algebra. Finally, we shall see  $L_j$  is contained in the center of  $L$ .

Suppose  $a$  and  $j(b)$  are arbitrary elements of  $L$  and  $L_j$  respectively. We calculate  $a \&^F j(b)$ .

$$\begin{aligned} a \&^F j(b) &= (a \vee j(b)^\perp) \wedge j(b) = [a \vee j(b)^\perp] \wedge j(b) \\ &= (a \wedge j(b)) \vee (j(b)^\perp \wedge j(b)) = a \wedge j(b). \end{aligned}$$

Since  $L_j$  is closed under complements and by the third property of the B-V.B. conditions. In a similar way, we can check  $j(b) \&^F a = j(b) \wedge a$ . Hence  $j(b)$  is in the center of  $L$ . □

*Remark 2.* In the proof of the proposition we used the inequality  $a^\perp \&a \leq a^\perp \&^F j(a)$ . Actually, it is easy to see that  $a \&b \leq a \&^F j(b)$  for arbitrary elements  $a, b$  in  $L$ . Moreover,  $j(L)$  is not only a Boolean subalgebra of the center of  $L$ , it is a complete Boolean subalgebra of the center of  $L$ .

We will see now the converse of this proposition.

**Proposition 9.** *Let  $L$  be an arbitrary complete orthomodular lattice. If  $Z_F(L)$  denotes the center of  $L$  and  $M$  is a complete subalgebra of  $L$  then the endomorphism  $j : L \rightarrow L$  defined by*

$$j_M(a) = \bigwedge \{x \in M \mid a \leq x\}.$$

*satisfies the B-V.B. conditions and therefore  $L$  is an idempotent, right-sided quantale. The binary operation  $\&$  is given by  $a \&b = a \wedge j_M(b)$  where  $a, b$  are arbitrary elements of  $L$ .*

**Proof:** Clearly, given any element  $a$  of  $L$  we have:  $a \leq j_M(a)$  and  $j_M$  is idempotent. Moreover, if  $z \in M$  then  $j_M(z) = z$ ; using these results,  $j_M$  preserves order as the reader can check easily. Now, we shall see  $j_M$  preserves arbitrary suprema. Suppose  $\{a_i\}_{i \in I}$  is a family of elements of  $L$ . Since  $j_M$  preserves order, we only need to verify  $j_M(\bigvee_{i \in I} a_i) \leq \bigvee_{i \in I} j_M(a_i)$  but this is true since  $M$  is a complete subalgebra of  $Z_F(L)$  and therefore  $\bigvee_{i \in I} j_M(a_i) \in M$ .

It is not hard to prove the third and the fourth properties of the B-V.B. since  $j_M(a)$  is an element of the center of  $L$  and  $j_M$  preserves arbitrary joins. We only need to verify the second property; i.e.,  $j_M(a \wedge j_M(b)) = j_M(a) \wedge j_M(b)$ . Given

any element  $z$  of  $M$  and an arbitrary element  $a$  of  $L$  the identities hold

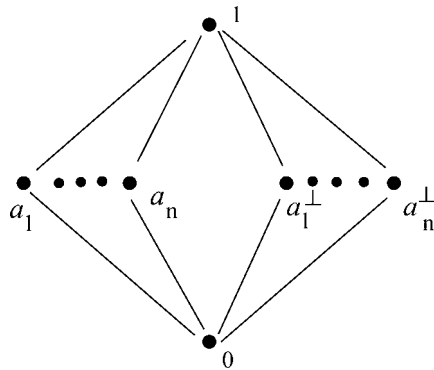
$$\begin{aligned} z \wedge j_M(a) &= \{z \wedge j_M(z \wedge a)\} \vee \{z \wedge j_M(z^\perp \wedge a)\} \\ &= \{z \wedge j_M(z \wedge a)\} = j_M(z \wedge a) \end{aligned}$$

In particular,  $j_M(a \wedge j_M(b)) = j_M(a) \wedge j_M(b)$ . Hence,  $j_M$  satisfies the B.-V.B. conditions as we claimed. □

We summarize the results in the following

**Theorem 1.** *Let  $L$  be a complete orthomodular lattice. There is a one to one correspondence between complete Boolean subalgebras  $M$  contained in the center of  $L$  and endomorphisms  $j : L \rightarrow L$  satisfying the Borceux–van den Bossche conditions.*

We would like to mention another example of an orthomodular lattice with trivial center. The lattice is called  $M_{O_n}$  for  $1 \leq n$ . In fact,  $M_{O_n}$  is not only orthomodular, it is modular but it is not distributive.



It is not hard to check that given  $i \neq j$ ,  $a_i \&^F a_j = a_j$  and  $a_j \&^F a_i = a_i$ . Hence, the center is trivial.

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